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19. Abstract

The research supported by this grant was essentially concerned with the numerical solution of the Zakai equation from nonlinear filtering. The Zakai equation is a linear stochastic equation of parabolic type and its solution provides a nonnormalized probability density from which we can reconstruct the solution of a dynamical system governed by stochastic ordinary differential equations when the observation is itself noisy. The Zakai equation presents some unique features in the sense that

(i) It has practical applications;
(ii) It is a highly advective, advention-diffusion linear parabolic equation;
(iii) The space dimension is very large (of the order of ten and more for practical applications);
(iv) It is a stochastic equation.

Each of the difficulties listed in (ii), (iii), (iv) leads to very challenging problems when it goes to the numerical solution, but their combination provides a formidable numerical problem which from our point of view requires computer power beyond the possibility of the existing hardwares and softwares.

In view of developing know now we have been investigating systematicc methods for solving the Zakai equation, which have proved reliable for solving problems with space dimension six at most (this number will increase with the progress of computers). The methodology that we have been studying relies on operator splitting methods in which one decouples, via time discretization, the stochastic part and the deterministic part of the Zakai equation. Using this approach we obtain subproblems which have either closed form solution (the stochastic part) or can be solved (the deterministic part; modulo the difficulty of the dimension) by efficient methods for traditional elliptic and parabolic problems. The same methodology can be extended to the parabolic inequalities of obstacle type originating from impulse control.

Concerning the space approximation itself it can be achieved by finite difference or finite element methods, with a specific treatment of the first order terms for which the first order upwinding methods investigated by some authors are by far too dissipative. We have been therefore investigating second order upwinding methods which are by far much less dissipative (in the detailed final report we also comment on third order upwinding methods).

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Numerical Methods for Parabolic Equations and Inequalities in Very High Dimensions

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1. Introduction. Motivation

The main goal of this projects was to investigate the numerical solution of *parabolic equations in very high dimension*, the principal motivation being the solution of the so-called *Zakai equation* from nonlinear filtering. As it will be seen in Section 2, below, this equation is of the *advection-diffusion* type and therefore is closely related to other equations of this type such as the Navier-Stokes equations from Fluid Dynamics; since these equations are also of interest for the Department of Defense some aspects of their numerical solution, byproducts of the present investigation will be considered in the present document.

The content of this report is the following:

In Section 2, we describe the Zakai equation and its origin. In Section 3, we consider the solution of time dependent problems by operator splitting methods, and show in Sections 4, 5, 6 how these methods apply to the Zakai equation, to parabolic inequalities of obstacle type and to Navier-Stokes equations.

In Section 7 we comment on the solution of advection-diffusion problems by upwinding, particle and characteristics methods. In Section 8 we go back to the Zakai equation and its stochastic dynamical system origin and comment about the feasibility of the Zakai equation approach for the analysis of such systems. (KR)

2. The Zakai equation.

We consider a *signal process* given by

(2.1) $$dx_t = b(t, x_t)dt + \sigma(t, x_t)dw_t.$$

Here $t\in[0,T]$, $x\in\mathbb{R}^d$ and x_0 has a given probability density $p_0(x)$. The vector $w_t=\left\{w_t^d, \ldots, w_t^d\right\}$ is a d-dimensional Brownian motion defined on a probability space (Ω, F, p^*). Also, $b(t, x) = \left\{b_i(t, x)\right\}_{i=1}^{d}$ and $\sigma(t, x)= (\sigma_{ij}(t,x))$, $i,j=1, \ldots, d$, have components defined on $[0,T]\times\mathbb{R}^d$ which are bounded and smooth. We suppose now that the observation is given by

$$dy_t= h(t, x_t) + g(t)dw_t + \tilde{g}(t)d\tilde{w}_t. \tag{2.2}$$

Here, $y\in\mathbb{R}^m$, $y_0=0$ and $\tilde{w}_t=\left\{\tilde{w}_t^j\right\}_{j=1}^{m}$ is a m dimensional Brownian motion defined on (Ω, F, p^*) and is independent of w. The function $h(t, x) = \left\{h_j(t, x)\right\}_{j=1}^{m}$ has essentially the same smoothness and boundedness properties than b and σ. We shall suppose that matrices $g(t) = (g_{ij}(t))$, $i=1, \ldots m$, $j=1, \ldots d$, and $\tilde{g}(t)=(\tilde{g}_{kl}(t))$, $k, l=1, \ldots m$ are continuous on $[0,T]$. We suppose that

$$g(t)g(t)^T+\tilde{g}(t)\tilde{g}(t)^T= I. \tag{2.3}$$

Condition (2.3) is a normalization hypothesis which can always be satisfied by re-scaling the observation process.

Denoting by g_j the j-th row of g we define the vector c_j by

$$c_j(t) = \sigma(t)\, g_j^T(t),$$

and then $\bar{h}_j$ by

$$\bar{h}_j(t) = h_j(t) - \operatorname{div} c_j(t), \quad j=1, \ldots d.$$

Define now an operator H_j, mapping $H^1(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$, by

$$H_j(t)\, u(t) = \bar{h}_j(t)\, u(t) - (C_j(t), \nabla u(t)),$$

for $u \in H^1(\mathbb{R}^d)$.

The elliptic operator L^* is the formal adjoint of the generator of (2.1), i.e.

$$L^*(t) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(t) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} b_i(t);$$

here $a= \left\{a_{ij}\right\} =\sigma\sigma^T$.

Define $Y_t = \sigma\left\{y_s:\ 0 \le s \le t\right\}$ for the σ-field generated by the observations up to time t. There exists an unnormalized density q_t of x_t conditioned on Y_t; q_t satisfies

$$q_0 = p_0 \tag{2.4}$$

and the following stochastic partial differential equation

$$q_t(x) = p_0(x) + \int_0^t L_s^* \, q_s(x)\, ds + \sum_{j=1}^{m} \int_0^t H_j(s)\, q_s(x)\, dy_s^j, \tag{2.5}$$

for $t \in [0,T]$. Equation (2.5) is precisely the Zakai equation (see, e.g., [5], [10] for more details).

3. Operator Splitting Methods for Time Dependent Problems.

Let's consider a (deterministic) dynamical system modelled by

$$\frac{du}{dt} + A(u) = 0, \text{ for } t>0, \tag{3.1}$$

$$u(0) = u_0. \tag{3.2}$$

In (3.1), (3.2), A is a(possibly nonlinear) operator from a Hilbert space H into itself, and $u_0 \in H$. We suppose that A has a nontrivial decomposition of the following type

$$A = A_1 + A_2; \tag{3.3}$$

by nontrivial we mean that A_1, A_2 are individually simpler than A (indeed A_1 and/or A_2 can be multivalued operators).

There exist many schemes taking advantage of decomposition (3.3) (see, e.g., [1], [2], [3] for such schemes). We shall present here some of them:

3.1 Fractional Step Schemes

1st. Scheme:

(3.4) $$u^0 = u_0;$$

then for $n \geq 0$, u^n *being known we compute* $u^{n+1/2}$ *and then* u^{n+1} *as follows*

(3.5) $$\frac{u^{n+1/2} - u^n}{\Delta t} + A_1(u^{n+1/2}) = 0,$$

(3.6) $$\frac{u^{n+1} - u^{n+1/2}}{\Delta t} + A_2(u^n) = 0.$$

This scheme is at most first order accurate in Δt. To obtain a better accuracy, we can "symmetrize" scheme (3.4) - (3.6) by using the following scheme

2nd Scheme:

(3.7) $$\frac{u^{n+1/3} - u^n}{\Delta t/2} + A_1\,(u^{n+1/3}) = 0,$$

(3.8) $$\frac{u^{n+2/3} - u^{n+1/3}}{\Delta t} + A_2\,(u^{n+2/3}) = 0,$$

(3.9) $$\frac{u^{n+1} - u^{n+2/3}}{\Delta t/2} + A_1\,(u^{n+1}) = 0.$$

It follows from, e.g., [3] (see also [4]), that in many situations, the above scheme is second order accurate with respect to Δt.

3.2. Alternating Direction Schemes.

The above schemes are not too well suited to capture steady state solutions of (3.1), i.e., solutions of

(3.10) $$A(u) = 0$$

(if such solutions exist). This drawback is corrected if one uses the schemes below:

Peaceman-Rachford Scheme:

(3.10) $u^0 = u_0;$

then for $n \geq 0$,

(3.11) $$\frac{u^{n+1/2} - u^n}{\Delta t/2} + A_1 (u^{n+1/2}) + A_2(u^n) = 0,$$

(3.12) $$\frac{u^{n+1} - u^{n+1/2}}{\Delta t/2} + A_1 (u^{n+1/2}) + A_2 (u^{n+1}) = 0.$$

Douglas-Rachford Scheme:

(3.13) $u^0 = u_0;$

then for $n \geq 0$,

(3.14) $$\frac{u^{n+1/2} - u^n}{\Delta t} + A_1 (u^{n+1/2}) + A_2(u^n) = 0,$$

(3.15) $$\frac{u^{n+1} - u^n}{\Delta t} + A_1 (u^{n+1/2}) + A_2 (u^{n+1}) = 0.$$

θ-Scheme: With $\theta \in (0,.5)$, this scheme is defined by (3.13) and

(3.16) $$\frac{u^{n+\theta} - u^n}{\theta \Delta t} + A_1 (u^{n+\theta}) + A_2(u^n) = 0,$$

(3.17) $$\frac{u^{n+1-\theta} - u^{n+\theta}}{(1\text{-}2\theta)} + A_1 (u^{n+\theta}) + A_2(u^{n+1-\theta}) = 0,$$

(3.14) $$\frac{u^{n+1} - u^{n+1-\theta}}{\theta \Delta t} + A_1 (u^{n+1}) + A_2(u^{n+1-\theta}) = 0.$$

All these schemes are first order accurate with respect to Δt in general, the last one being (for θ well chosen) better suited than the first two to capture steady states.

4. Application to the solution of the Zakai equation.

Using the notation of Section 2, we are considering the solution of

$$dy + A^*(t)y\, dt = B(t)\, y \cdot dw. \tag{4.1}$$

The idea here, is to consider

$$A^*(t)y\, dt - B(t)y \cdot dw \tag{4.2}$$

as the sum of two operators and using the approach associated to scheme (3.4) - (3.6), consider a sequence of problems of the form

$$d\varphi + A(t)\varphi\, dt = 0, \tag{4.3}$$

$$d\Psi = B(t)\Psi \cdot dw, \tag{4.4}$$

which are considerably simpler than (4.1); indeed, the φ equation is deterministic and the Ψ equation has a closed form solution.

The algorithm investigated has been analyzed in [5] and generalized to more complicated stochastic partial differential equations in [6]; it is described as follows:

Consider first (4.1) on the time interval [0,T] and define $\Delta t = k = T/(N+1)$. We shall define two processes y_{1k}, y_{2k} depending on k. We split therefore [0, T] in [0, k[, [k, 2k[, ... [Nk, (N+1)k[. Consider now [rk, (r+1)k[with r=0, 1, ..., N; then y_{1k}, y_{2k} are defined on this interval by the relations

$$dy_{1k} + A^*(t)\, y_{1k} dt = 0, \tag{4.5}$$

(4.6) $dy_{2k} = B(t)\, y_{2k} \cdot dw,$

(4.7) $y_{1k}(rk) = y_k^r;$

(4.8) $y_{2k}(rk) = y_k^{r+1/2}$

the sequences y_k^r, $y_k^{r+1/2}$ are defined as follows

(4.9) $y_k^{r+1/2} = y_{1k}\,((r+1)\,k-0),$

(4.10) $y_k^{r+1} = y_{2k}\Big((r+1)\,k-0\Big).$

Relations (4.5) - (4.10) define completely y_{1k}, y_{2k} on [0, T[. The processes y_{1k}, y_{2k} are right continuous and their discontinuity points are k, ... Nk (on [0, T[); we observe that (4.5) is *deterministic.*

The convergence of y_{1k}, y_{2k} to the solution y of (4.1) is proved in [5], justifying therefore the splitting approach.

In practice, problem (4.5) is an advection-diffusion problem whose solution will be discussed in Section 7.

5. Application to the solution of obstacle type parabolic inequalities.

It follows from [7], that applications in *impulse control*, lead to the solution of parabolic inequalities of the following type

(5.1) $\left(\frac{\partial u}{\partial t} + Au - f\right)(\Psi - u) = 0,$ *a. e. on* $\Omega \times (0,T),$

(5.2) $\frac{\partial u}{\partial t} + Au - f \leq 0,$ *a. e.*,

(5.3) $u - \Psi \leq 0,$ *a.e.*,

$$u = u_0 \text{ at } t = 0; \tag{5.4}$$

in (5.1) (5.4), A is an advection-diffusion elliptic operator, and Ψ defines the obstacle. From a numerical point of view, it may be more convenient to formulate (5.1) - (5.4) as a *parabolic variational inequality* such as

$$\int_\Omega \left(\frac{\partial u}{\partial t} + Au - f\right)(v - u)\, dx \geq 0, \ \forall v \in K(t), \tag{5.5}$$

$$u(t) \in K(t), \ \forall t \geq 0, \tag{5.6}$$

$$u(0) = u_0 \left(\in K(0)\right), \tag{5.7}$$

with

$$K(t) = \left\{v | v \in V, \ v(x,t) \leq \Psi(x,t) \ \text{a.e. on } \Omega \times (0,T)\right\}; \tag{5.8}$$

in (5.8), V is a Hilbert space of Sobolev type.

In (5.5) - (5.8) we have implicitely assumed that Ψ, and therefore the (convex) set K, vary continuously with t. Concerning the numerical solution of (5.5) - (5.8) by operator splitting methods, let's introduce

$$C(t) = \left\{v | v \in L^2(\Omega), \ v(x) \leq \Psi(x,t) \ \text{a.e. on } \Omega\right\}, \tag{5.9}$$

and the *indicator functional* I_C of C(t), defined by

$$I_C(v, t) = 0 \text{ if } v \in C(t), \ I_C(v,t) = +\infty \text{ if } v \notin C(t). \tag{5.10}$$

Problem (5.5) - (5.8) can then be formulated as a *nonlinear parabolic "equation"* (it is in fact a *multivalued* equation) by

$$0 \in \frac{\partial u}{\partial t} + Au + \partial I_C(u) - f, \tag{5.11}$$

$$u(0) = u_0; \tag{5.12}$$

in (5.11), $\partial I_C(u)$ is the *subdifferential* of the convex functional I_C at u; it is defined, $\forall v \in L^2(\Omega)$, by

(5.13) $$w \in \partial I_C(v,t) \Leftrightarrow I_C(z,t) - I_C(v,t) \geq \int_\Omega (z-v)\, w\, dx, \ \forall z \in L^2(\Omega).$$

Problem (5.11), (5.12) can be approximated by the following nonlinear parabolic equation (with $\varepsilon > 0$):

(5.14) $$\frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \Psi)^+ = f,$$

(5.15) $$u_\varepsilon(0) = u_0,$$

which corresponds to a *penalty* approximation of condition (5.6). Operator

(5.16) $$v \longrightarrow (v - \Psi)^+$$

is (for t given) a ("nice") *monotone* operator from $L^2(\Omega) \to L^2(\Omega)$.

Both problems (5.11), (5.12) and (5.14), (5.15) can be solved by operator splitting methods. For simplicity, we shall just describe the application of scheme (3.4)-(3.6), but the other schemes described in Section 3 also apply. We obtain then for problem (5.11), (5.12)

(5.17) $$u^0 = u_0;$$

then for $n \geq 0$, *we compute* $u^{n+1/2}$ *and* u^{n+1} *via*

(5.18) $$\frac{u^{n+1/2} - u^n}{\Delta t} + \partial I_C\left(u^{n+1/2}, (n+1)\Delta t\right) = 0,$$

(5.19) $$\frac{u^{n+1} - u^{n+1/2}}{\Delta t} + Au^{n+1} = f^{n+1}.$$

Problem (5.19) is an advection-diffusion elliptic problem whose solution is discussed in Section 7; on the other hand problem (5.18) has a *closed form* solution which is given by

(5.20) $$u^{n+1/2}(x) = \min\left(u^n(x), \Psi(x, (n+1)\Delta t)\right).$$

A similar conclusion would hold for problem (5.14), (5.15). In fact from the simplicity of scheme (5.17), (5.19) it seems that penalty is useless in this context; we have mentioned it however since it may be useful in some other situations. We can switch of course the splitting order and solve first the advection diffusion part, and then project as in (5.20).

6. Application to the Navier-Stokes equations.

Another advection-diffusion problem of interest is associated to the *Navier-Strokes equations* for viscous fluids. Concentrating on the *incompressible* case we obtain

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)\, u + \nabla p = f \quad \textit{in } \Omega. \tag{6.1}$$

$$\nabla \cdot u = 0 \quad \textit{in } \Omega \quad (\textit{incompressibility condition}), \tag{6.2}$$

$$u\,(x, 0) = u_0\,(x) \quad \textit{in } \Omega, \tag{6.3}$$

with appropriate boundary conditions; for simplicity we shall assume that

$$u = g \ \textit{on } \Gamma(=\partial\Omega). \tag{6.4}$$

In (6.1) - (6.4), $u = \{u_i\}_{i=1}^N$ is the *velocity*, p is the *pressure*, Ω is the *flow region*, $\nu(>0)$ is a *viscosity* coefficient and f a density of external forces and

$$(v \cdot \nabla) w = \left\{ \sum_{j=1}^{N} v_j \frac{\partial w_i}{\partial x_j} \right\}_{i=1}^{N}. \tag{6.5}$$

Following the approach in [8] (see also [9]) we shall discretize (6.1) - (6.4) by the following θ-scheme, which decouples *nonlinearity* and *incompressibility*:

$$u^o = u_o, \tag{6.6}$$

and then for $n \geq 0$,

$$\frac{u^{n+\theta} - u^n}{\theta \Delta t} - \alpha \nu \Delta u^{n+\theta} + \nabla p^{n+\theta} = f^{n+\theta} + \beta \nu \Delta u^n - (u^n \cdot \nabla) u^n \quad \textit{in } \Omega, \tag{6.7$_1$}$$

$$(6.7)_2 \quad \nabla\cdot u^{n+\theta} = 0 \quad in\ \Omega,$$

$$(6.7)_3 \quad u^{n+\theta} = g^{n+\theta} \quad on\ \Gamma,$$

$$(6.8)_1 \quad \frac{u^{n+1-\theta}-u^{n+\theta}}{(1\text{-}2\theta)\Delta t} - \beta\nu\Delta u^{n+1-\theta} + (u^{n+1-\theta}\cdot\nabla)\, u^{n+1-\theta} = f^{n+1-\theta} + \alpha\nu\Delta u^{n+\theta} - \nabla p^{n+\theta} \quad in\ \Omega,$$

$$(6.8)_2 \quad u^{n+1-\theta} = g^{n+1-\theta} \quad on\ \Gamma,$$

$$(6.9)_1 \quad \frac{u^{n+1}-u^{n+1-\theta}}{\theta\Delta t} - \alpha\nu\Delta u^{n+1} + \nabla p^{n+1} = f^{n+1} + \beta\nu\Delta u^{n+1-\theta} - (u^{n+1-\theta}\cdot\nabla)u^{n+1-\theta} \quad in\ \Omega,$$

$$(6.9)_2 \quad \nabla\cdot u^{n+1} = 0 \ in\ \Omega,$$

$$(6.9)_3 \quad u^{n+1} = g^{n+1} \quad on\ \Gamma.$$

A good choice for θ is $\theta = 1\text{-}1/\sqrt{2}$ and for α and β

$$\alpha = \frac{\theta}{1\text{-}\theta}, \qquad \beta = \frac{1\text{-}2\theta}{1\text{-}\theta}.$$

Numerical results obtained using the above scheme, combined to finite element methods are given in [8]; the above methods has been extended to two-phase flow problems in [9].

7. Numerical Solution of Advection - Diffusion Problems

7.1. Synopsis. Motivation

We follow here the presentation in [8], completed by some recent findings concerning upwinding methods of order three. In order to concentrate on Zakai equation related problems we shall illustrate the subsequent developments by considering an advection-diffusion problem associated to the following *stochastic ordinary differential equation*

(7.1) $\qquad dx = V(x)\,dt + dw,$

where x is an N-dimensional vector and w a noise. Assuming convenient hypotheses on the noise w, the probability of finding at time t the state vector x, in neighborhood of $\xi \in \mathbb{R}^N$ of measure $d\xi = d\xi_1 \ldots d\xi_N$, is $p(\xi,t)\,d\xi$ where

$$p(\xi,t) = \frac{q(\xi,t)}{\int_{\mathbb{R}^N} q(\eta,t)\,d\eta}, \tag{7.2}$$

where the unnormilizaed probability density q satisfies a parabolic equation (see Section 2 for details). In the particular case where V is *divergence free* and for simple noise models, this parabolic equation reduces to

$$\frac{\partial q}{\partial t} - \varepsilon\,\nabla^2 q + V \cdot \nabla p = 0. \tag{7.3}$$

An interesting and difficult case is the one where the level of noise is "weak", implying that ε is "small". We have then an *advection dominated advection-diffusion equation.*

Using the splitting methods described in Section 4, the solution of such equations plays a fundamental role in the implementation of some solution methods for the Zakai equation.

In the sequel, we consider the following *initial boundary value problem.*

$$\frac{\partial p}{\partial t} - \varepsilon \nabla^2 p + V \cdot \nabla p = f \text{ in } \Omega \times (0,T), \tag{7.4$_1$}$$

$$p = g \textit{ on } \partial\Omega \times (0,T), \tag{7.4$_2$}$$

$$p(x,0) = p_0(x) \textit{ in } \Omega, \tag{7.4$_3$}$$

with $\Omega \subset \mathbb{R}^N$.

For solving such problems, particularly the ones associated to filtering, one has to face two outstanding difficulties, namely

(i) When ε is small, the problem is ***advection-dominated***,

(ii) For practical problems, we have N>3 (in fact we may have N~10 and more).

The methods described here have been applied in [8] to N=2 to 6.

7.2 Solution of problem (7.4) by finite differences and upwinding methods.

The methods we need have to be *accurate* and *robust*; in our opinion, *we have to avoid those methods requiring parameter tuning.*

For simplicity, we consider the case where $\Omega = (0,1)^N$, with N=2; the extension to N>2 is straight forward. With I a *positive integer*, we define h by h=1/I+1 and consider over $\bar{\Omega}=\Omega\cup\partial\Omega$ the mesh points

$$M_{ij} = \Big\{ih, jh\Big\}; \quad 0 \leq i,j \leq I+1. \tag{7.5}$$

At the points M_{ij} interior to Ω (i.e., $1\leq i,j \leq I$) we approximate (7.4) by the following *finite difference scheme* (with $V=\{V_1,V_2\}$):

$$(7.6)_1 \quad \left\{ \begin{aligned} &\frac{p_{ij}^{n+1} - p_{ij}^n}{\Delta t} - \varepsilon \frac{p_{i+1j}^{n+1} + p_{i-1j}^{n+1} + p_{ij+1}^{n+1} + p_{ij-1}^{n+1} - 4p_{ij}^{n+1}}{h^2} \\ &+ V_1^+ (M_{ij}) \frac{p_{ij}^{n+1} - p_{i-1j}^{n+1}}{h} - V_1^- (M_{ij}) \frac{p_{i+1j}^{n+1} - p_{ij}^{n+1}}{h} \\ &+ V_2^+ (M_{ij}) \frac{p_{ij}^{n+1} - p_{ij-1}^{n+1}}{h} - V_1^- (M_{ij}) \frac{p_{ij+1}^{n+1} - p_{ij}^{n+1}}{h} \\ &= f(M_{ij}, (n+1)\Delta t), \end{aligned} \right.$$

with , in (7.6), $\Delta t(>0)$ a *time discretization step*, $p_{ij}^n \sim p(M_{ij}, n\Delta t)$, at = max (0,a), a^+ = max (0,-a), $\forall a \in \mathbb{R}$, and

$$(7.6)_2 \qquad p_{kl}^{n+1} = g(M_{kl}, (n+1)\,\Delta t) \text{ if } M_{kl} \in \partial\Omega,$$

$(7.6)_3 \qquad p^0_{kl} = p_0(M_{kl}).$

Scheme (7.6) is of the *backward Euler* type for the *time discretization* and of the *first order upwinded type* for the *space discretization*. Probabilists favor the above finite difference scheme because it satisfies a *discrete maximum principle* and therefore possesses a *probabilistic interpretation*. Unfortunately the above scheme is only *first order accurate*, *very dissipative* and not well-suited for those situations where ε is small and V has a fast variation over Ω.

An interesting alternative to (7.6) is obtained through a space/time discretization which is second order accurate and also of the upwind type (however, it does not satisfy the discrete maximum principle). Such a scheme is obtained as follows:

$(7.7)_1 \qquad p^0_{kl} = p_0(M_{kl}),\ p^1_{kl}$ *is obtained (for example) via* (7.6);

then for $n \geq 1$ *and* $2 \leq i,j \leq I-1$, *discretize* $(7.4)_1$ *by*

$$(7.7)_2 \quad \begin{cases} \dfrac{\frac{3}{2}\,p^{n+1}_{ij} - 2\,p^{n}_{ij} + \frac{1}{2}\,p^{n-1}_{ij}}{\Delta t} - \varepsilon\,\dfrac{p^{n+1}_{i+1j} + p^{n+1}_{i-1j} + p^{n+1}_{ij+1} + p^{n+1}_{ij-1} - 4p^{n+1}_{ij}}{h^2} \\[2ex] + V^+_1(M_{ij})\dfrac{\frac{3}{2}\,p^{n+1}_{ij} - 2p^{n+1}_{i-1j} + \frac{1}{2}p^{n+1}_{i-2j}}{h} + V^-_1(M_{ij})\dfrac{\frac{3}{2}\,p^{n+1}_{ij} - 2p^{n+1}_{i+1j} + \frac{1}{2}p^{n+1}_{i+2j}}{h} \\[2ex] + V^+_2\,(M_{ij})\,\dfrac{\frac{3}{2}p^{n+1}_{ij} - 2p^{n+1}_{ij-1} + \frac{1}{2}p^{n+1}_{ij-2}}{h} + V^-_2\,(M_{ij})\,\dfrac{\frac{3}{2}p^{n+1}_{ij} - 2p^{n+1}_{ij+1} + \frac{1}{2}p^{n+1}_{ij+2}}{h} \\[2ex] = f(M_{ij}, (n+1)\Delta t). \end{cases}$$

If $M_{ij} \in \Omega$ with either $M_{ij\pm1}$ or $M_{i\pm1j}$ belonging to Γ, it is possible that $M_{ij\pm2}$ or $M_{i\pm2j}$ are outside $\overline{\Omega}$; in such a case we can use to discretize $V \cdot \nabla p$ at M_{ij}, a first order scheme like in $(7.6)_1$, or alternatively a *centered second order* approximation like

$$(7.8) \qquad (V \cdot \nabla p)\,(M_{ij}) \sim V_1(M_{ij})\,\frac{p_{i+j} - p_{i-1j}}{2h} + V_2(M_{ij})\,\frac{p_{ij+1} - p_{ij-1}}{2h}$$

The boundary conditions are treated as in $(7.6)_2$. The fact that the problems under consideration may have a fast dynamics, requires the use of small h and Δt; indeed, we can take advantage of the fact that Δt is small, to solve the above discrete elliptic problems by successive over-relaxation, since the method has good vectorization and parallelization properties (in practice few iterations will insure convergence at each time step).

Numerical experiments (see, e.g., [8]) show the superiority of the second order upwinding method, over the first order method (it is more accurate, less dissipative and almost as easy to implement).

7.3. A third order in space upwinded scheme.

Influenced by third order in space, upwinded finite difference schemes, recently developed in Japan, for Computational Fluid Dynamics purposes, we shall describe here a variant of scheme (7.7) which uses a third order accurate space discretization of the advection and which is second order accurate in time. The basic principle of this scheme is to combine two second order accurate schemes, one centered (in fact, it is scheme (7.8)) and the one used in (7.7). The resulting scheme is *more accurate* than scheme (7.7) but *less robust*; we have then $(7.7)_1$ and

$$(7.9)\quad \begin{cases} \dfrac{\frac{3}{2}\,p^{n+1}_{ij} - 2\,p^{n}_{ij} + \frac{1}{2}\,p^{n-1}_{ij}}{\Delta t} - \varepsilon\,\dfrac{p^{n+1}_{i+1j} + p^{n+1}_{i-1j} + p^{n+1}_{ij+1} + p^{n+1}_{ij-1} - 4p^{n+1}_{ij}}{h^2} + \\[2ex] V_1^+(M_{ij})\dfrac{\frac{1}{3}p^{n+1}_{i+1j} + \frac{1}{2}p^{n+1}_{ij} - p^{n+1}_{i-1j} + \frac{1}{6}p^{n+1}_{i-2j}}{h} + V_1^-(M_{ij})\dfrac{\frac{1}{3}\,p^{n+1}_{i-1j} + \frac{1}{2}p^{n+1}_{ij} - p^{n+1}_{i+1j} + \frac{1}{6}p^{n+1}_{i+2j}}{h} \\[2ex] +\ V_2^+\,(M_{ij})\dfrac{\frac{1}{3}p^{n+1}_{ij+1} + \frac{1}{2}p^{n+1}_{ij} - p^{n+1}_{ij-1} + \frac{1}{6}p^{n+1}_{ij-2}}{h} + V_2^-(M_{ij})\dfrac{\frac{1}{3}p^{n+1}_{ij-1} + \frac{1}{2}p^{n+1}_{ij} - p^{n+1}_{ij+1} + \frac{1}{6}p^{n+1}_{ij+2}}{h} \\[2ex] = f(M_{ij},\,(n+1)\Delta t); \end{cases}$$

we shall use (7.8) for those grid points at distance h of the boundary.

8. Further Comments. Conclusion

Operator splitting methods provide a systematic way to obtain efficient solution methods for parabolic equations and inequalities in high dimension. This point of view has been illustrated by the design of efficient methods for fundamental problems such as the Zakai equation, the Navier-Stokes equations and time dependent obstacle problems (other applications, like *liquid crystal* problems are discussed in [8]). Concerning now more specifically the Zakai equation we would like to point out the following facts:

(a) Due to the very high space dimension associated to practical problems, there is no hope that fast solution can be achieved for several years to come. Indeed the methods described here are quite fast, since the experiments in [8] show that the solution cost per time step is of the order of the number of grid points; however this number is so large that the task is still formidable, requiring among other things highly massive parallelism and huge computer memory.

(b) It is perplexing to realize that several partial differential equation specialists advocate to solve deterministic systems of advection-diffusion type (such as the Navier-Stokes equations) by *particle methods* where the advection is treated via the integration of a (large) system of ordinary differential equations, and the diffusion by a random walk method. Indeed the stochastic dynamical systems leading to the Zakai equation are already in the above form since they couple the system of ordinary differential equations modelling the dynamic of the problem to a noise "generator", and this suggests therefore to deal more directly with the original form of the problem.

(c) Since the Zakai equation is linear even if the associated dynamical system is not, we can see it as a simplication. This is a quite superficial conclusion since in the case of a treatment by particle or characteristic methods we shall have nevertheless to solve a very large number of nonlinear ordinary differential systems very close to the original one in order to accurately construct the characteristic curves associated to the advection term.

Our very conclusion concerning the Zakai equation will be therefore that due to lack of computing power it is hopeless that genuine real life systems can be treated by this approach; (at least for several years to come) in fact the situation is very close to the one encountered in the solution of the time dependent Schrodinger equation where only the simplest molecules can be investigated, precisely due to the dimension difficulty.

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